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Dynamical quantum localization

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Abstract

Quantum localization is essential for quantum measurement. This paper presents a deterministic dynamics of quantum localization, in which the hidden variables are the phase factors of matter waves, which replace the intrinsic stochastic variables of quantum state diffusion. In its semiclassical form, the stochasticity of quantum measurement comes from the chaotic Hamiltonian dynamics of the corresponding classical system.

1 Introduction

Newton discovered a classical deterministic dynamical model of the Solar system, which was used to predict the future state of the system from its present state. Apart from relativistic corrections, it is still used today. Schrödinger's equation provides a deterministic dynamical model of atomic systems, apart from relativistic corrections. The Copenhagen interpretation divides the world into classical and quantum domains, each apparently obeying its own deterministic dynamical laws, separated by an indeterminate boundary, where the stochasticity of quantum measurement appears. Several years before Copenhagen, Schrödinger tried to represent classical degrees of freedom as wave-packet solutions of his equation, but Lorentz used Schrödinger's equation to show that the spreading of the wave packets would increase without limit, so that small classical systems could not remain localized, forcing Schrödinger to abandon the idea.

In reading this paper, it helps to take Schrödinger's point of view as it was during his dialogue with Lorentz, since it addresses the problem of localization that faced him at that time.

The Copenhagen interpretation can be used to predict the probability of the result of a quantum measurement, but it is not a dynamical theory, provides no consistent dynamical model and is not deterministic. Schrödinger wanted a single dynamics for quantum systems, classical systems and the interaction between them, a universal dynamics of the material world known at his time, like Newton's classical dynamics of the known material world of *his* time. Matter waves disperse, spreading out with time, but classical particles do not. The waves can only represent the particles if something keeps them localized. This is the localization problem of quantum theory, which is simpler than the measurement problem. Its solution leads in principle to the solution of the measurement problem, although in practice this is complicated because the quantum dynamics of measuring apparatus is complicated.

If both classical and quantum systems are to obey the laws of quantum dynamics, then Schrödinger's equation must be modified to produce localization.

QSD: Quantum state diffusion

Of the many attempts to solve the localization problem, the closest to Schrödinger's original idea is continuous spontaneous localization, using a stochastic Schrödinger equation or quantum state diffusion (QSD). These are equivalent theories in which a stochastic term is added to Schrödinger's original deterministic equation. QSD is introduced in [1], together with a description of early work, numerical methods and applications. A quantum measurement of the dynamical variable X , leads to the continuous stochastic relaxation of the time-dependent matter wave toward one of the measured eigenstates of X , with probabilities given by the

Born rules. For sufficiently massive particles, it leads to the required localization, where the localized wave packet obeys Newton's laws with a classical Hamiltonian.

For a two-state system, the state vector can be represented by a point on the Bloch sphere, with two orthogonal states $|+\rangle$ and $|-\rangle$ corresponding to the N and S poles.

Quantum state diffusion is a diffusion of a quantum state in the space of quantum states. Contrary to uniform diffusion of particles, which spreads them into bigger regions, the rate of diffusion is variable and the state diffuses toward one or other of the eigenstates of σ_z , so the quantum state tends to be concentrated more and more in a small area around the eigenstates. The final probabilities obey the Born rules.

According to QSD, both measurement and localization are physical processes that take a finite time. This localization competes with the dispersion or delocalization due to the Hamiltonian. There are no quantum jumps.

Localization in one dimension

For our purposes it is convenient to consider motion in one dimension with a continuous dynamical variable X , which might be a component of either position x or a momentum p . Δt is a finite interval of time, which tends to zero at the end of the analysis, and all powers of Δt higher than the first are neglected. Before the limit is taken, wave functions are defined only at the times $N\Delta t$, with integral N . The stochastic function $\Delta\xi(t)$ has independent values at these times, and its mean and variance at each time are

$$M\Delta\xi(t) = 0, \quad M(\Delta\xi(t))^2 = \Delta t$$

where M is the mean. Notice that $\Delta\xi(t)$ is of order $(\Delta t)^{\frac{1}{2}}$. No other properties of $\Delta\xi(t)$ are needed. After N time intervals the fluctuations built up as $N^{\frac{1}{2}}\Delta t$, which is proportional to the square root of the time. This procedure is equivalent to the Ito calculus. For simplicity, we do not include normalization corrections, for which see [1].

Let H be the Hamiltonian, and suppose that X is to be localized. The standard form for the increment in the wave function ψ is then

$$\Delta\psi = -\frac{i}{\hbar}H\psi\Delta t + \gamma_0(X - \langle\psi|X|\psi\rangle)\psi\Delta\xi,$$

where the second term is the important nonlinear stochastic localization term, whose effect is negligible for small quantum systems, but dominant for classical degrees of freedom. We consider the latter case for short periods of time, for which the Hamiltonian can be neglected, leaving only the stochastic localization term, with γ_0 as a rate constant for localization. Details are given in this notation in [1].

The result is a diffusion of the matter wave in the space of matter waves. For illustration consider the very special initial state

$$\psi(X, 0) = a\delta(X - X_0) + b\delta(X - X_1) \quad \text{where} \quad |a|^2 + |b|^2 = 1.$$

The system remains in the two-dimensional space spanned by the delta-functions, and its state can be represented by a point on the Bloch sphere. As for the two-state system, the state localizes to one of the two delta-functions.

Note that the rate of diffusion depends on a and b , and is zero for the two poles of the sphere, when either a or b is zero. Hence the system localizes toward $X = X_0$ or $X = X_1$. For an initial state (a, b) , the probability of reaching the N pole is $|a|^2$, according to the Born rules.

According to QSD, localization is a continuous process that takes a finite time. There are no quantum jumps in zero time. But current experiments can only access the initial unlocalized state and the final localized state. They can also put very wide bounds on the value of γ_0 . Further details of the localization process are not accessible to current experiments. Consequently there are many different ways of formulating X -localization that are consistent with experiment.

For example we could put

$$\Delta\psi = \gamma_0(f(\mathbf{X}, t) - \langle\psi|f(\mathbf{X}, t)|\psi\rangle)\psi\Delta\xi$$

with a choice of many different functions $f(X, t)$ and different γ_0 , or even

$$\Delta\psi = \gamma_0(\Delta\xi(\mathbf{X}, t) - \langle\psi|\Delta\xi(\mathbf{X}, t)|\psi\rangle)\psi,$$

where $\Delta\xi(X, t)$ is a smooth function of the continuous variable X for fixed t and a stochastic function of the discrete variable t for fixed X , which satisfies the conditions

$$M(\Delta\xi(\mathbf{X}, t) - \langle\psi|\Delta\xi(\mathbf{X}, t)|\psi\rangle) = 0, \quad M(|\Delta\xi(\mathbf{X}, t)|^2) = \Delta t.$$

Note that if Δt is non-zero, but sufficiently small, then the correction terms with higher powers of Δt than the first may be negligible for many applications. This is what we need for deterministic localization dynamics.

Dynamical localization

Suppose at first that the matter wave has the form

$$\psi(X, t) = D(X, t)^{\frac{1}{2}} e^{i\eta(X, t)},$$

so that

$$e^{2i\eta(X, t)} = \psi(X, t)/\psi^*(X, t),$$

where D is a real non-negative density in X , η is a real phase, and both are sufficiently smooth functions of their arguments. The dynamical theory is obtained by replacing the stochastic variable $\Delta\xi$ by a $\Delta\xi$ obtained from the phase of the matter wave: We assume that the fluctuation is proportional to the phase factor:

$$\Delta\xi = \Delta t^{\frac{1}{2}} e^{2i\eta}.$$

The change in ψ in the time interval Δt is then

$$\Delta\psi(X, t) = \gamma_0(\Delta\xi(X, t) - \langle\psi|\Delta\xi(X, t)|\psi\rangle)\psi(X, t)$$

with the new definition of $\Delta\xi(X, t)$.

Quasiclassical theory

The semiclassical dynamical theory of localization can be taken further by using the properties of the classical motion. For systems of few freedoms, with constant or periodic Hamiltonians, the motion is qualitatively distinct in the regular and chaotic regions of phase space. $D(X, t)$ is a classical density function in X -space, $S(X, t)$ is a solution of the time-dependent Hamilton-Jacobi equation, and the quantum phase is given by

$$\eta(X, t) = S(X, t)/\hbar.$$

For chaotic motion the coordinates and momenta of a trajectory are stochastic functions, with a finite autocorrelation time $\Delta_{ac}t$ determined by the Liapunov exponent of the motion. The classical action function $S(x, t)$ is also stochastic, and so is the phase factor above. For time intervals longer than $\Delta_{ac}t$, the phase factors are statistically independent, so there is quantum state diffusion, and all that this implies. Let t_{ho} be a time beyond which the higher-order corrections to the QSD equation become significant. Then there will be quantum state diffusion if

$$t_{ho} \gg \Delta_{ac}t,$$

so that a value of Δt can be chosen to satisfy both the autocorrelation and the higher-order conditions. For *regular* classical motion, the action function is not stochastic, and there is no guarantee of localization.

There are other theories for which the stochastic properties of quantum measurement have been related to classical chaos, for example Bohmian theories [2,3], and the work of Geszti [4].

5 Conclusions

According to this dynamical theory of localization:

- 1 The ‘hidden variables’ of quantum localization and measurement are the phases of matter waves.
- 2 Semiclassical mechanics can be used. Chaos is important.
- 3 The stochastic properties and the Born rules are due to chaotic motion of the corresponding classical system. Localization in a quantum measurement proceeds following the Born rules, as for QSD.
- 4 If we could get access to the regular region experimentally, the Born rules would not apply.
- 5 All ‘classical’ degrees of freedom are already localized. They were localized early in the history of the universe.

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References

1. I. C. Percival. *Quantum State Diffusion*. (Cambridge University Press, Cambridge, England, 1998)
2. D. Bohm and B. J. Hiley. *The Undivided Universe*. (Routledge, London, 1993)
3. P. R. Holland. *The Quantum Theory of Motion*. (Cambridge University Press, Cambridge, England, 1993)
4. T. Geszti. Gravitational self-localization in quantum measurement. *Phys. Rev. A* **69**, 032110 (2004)